Applied Mathematics and Computation 353 (2019) 254–264

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Forced waves and their asymptotics in a Lotka–Volterra cooperative model under climate change☆

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ARTICLE INFO

MSC:
35K57
35C07
92D40
92D25

Keywords:
Lotka–Volterra cooperative system
Climate change
Heterogeneity
Forced wave front

ABSTRACT

The existence and asymptotics of forced traveling wave front in a Lotka–Volterra cooperative system under climate change are concerned in this paper. By constructing appropriate upper and lower solutions combined with the monotone iteration scheme, we show that for any given positive speed of the shifting habitat edge, there exists a nondecreasing wave front with the speed consistent with the habitat shifting speed, which indicates that no matter how moderate of the habitat shifts, two species will still be driven to extinction as the mutualistic effects exist but weak. Numerical simulations are also given to illustrate our results.

1. Introduction

Climate change such as global warming is believed to be the greatest threat to biodiversity [31]. Those species that fail to respond effectively to warming or habitat loss have been undergoing a high risk of extinction [39]. A recent report [40] illustrated that if there was a 4.5°C global mean temperature increase, then 69% of plant species would forfeit in Amazon, 89% of amphibians would turn into locally extinct in south-west Australia, up to 90% of amphibians, 86% of birds and 80% of mammals would potentially die out in the Miombo Woodlands and Southern Africa. In the wake of global warming, species are moving from their natural habitats and intend to migrate towards the poles and up elevation where is cooler. For example, umbrella trees are creeping northwards through the United States [29], tropical birds in New Guinean mountains are migrating upslope rapidly [8] and southern Africa's typical quiver tree is escaping from the equator [26]. Climate change drives the shifts in species range and distribution [4,25].

One critical question is whether a given species is capable of moving along with the continuously changing climate. For birds and certain mammals, there is at least an opportunity that they can keep pace with their ideal habitat as it shifts toward the poles or into higher altitudes. For reptiles and plants, it becomes improbable. And for those species already living in the poles-like places such as Arctic, there may be nowhere further north to go.

To characterize the heterogeneity of the shifting habitat, one effective way is to consider the environment "on the move" induced by the global climate change. Since the boundary between suitable and unsuitable habitat varies, we can take into account the favourable habitats move in the positive direction at a fixed speed of c units per timestep. In order to investigate

☆ This work was supported by NSF of China [Grant nos. 11401096, 61872396 and 11601384].
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https://doi.org/10.1016/j.amc.2019.01.058
0096-3003/

the patch moving on the persistence of a single species, Berestycki et al. [1] incorporated the climate shifts as well as the phenotypic characteristics of the population, such as dispersal ability, intrinsic growth rate, mutations and intraspecific competitions into Species Distribution Models and introduced a simplified model, taking the form of a reaction-diffusion equation

$$u_t(t,x) = du_{xx}(t,x) + g\big(x-ct, u(t,x)\big), \qquad t>0, x\in\mathbb{R}. \tag{1}$$

Here $u(t, x)$ denotes the population density at time t and location x. The function g represents the net effect of reproduction and mortality and $d>0$ is the diffusion rate for species. Since the climate change drives the space to become heterogeneous, some regions are more favourable than others for the species. The growth function is therefore heterogeneous but not periodic (see [43] and references therein) in space, and the term $g(x-ct, u(t,x))$ expresses the suitability profile of this habitat. They deduced a critical size of the favorable patch for species persistence and showed a striking asymmetry in the comoving population profile by using analytical approaches.

A traveling wave front is able to describe a species that was well distributed over some certain area would gradually expand its range to the whole environment. Global warming tendency offers chances for species to expand their ranges northwards. If the species can track the climatic change, what happens to the size and form of its population profile as time evolves? How the heterogeneities induced by climate change influence the characteristics of front propagation such as front speeds, front profiles, and front locations [20,44]? To comprehend the effect of range shifts on genetic diversity, Garnier and Lewis [10] studied the existence of traveling wave fronts to the population model (1). They decomposed the traveling wave solution into neutral genetic components and then analysed the spatio-temporal dynamics of its genetic structure. More recently, Berestycki and Fang [2] studied the complete existence and multiplicity of forced waves to (1) when $g(s, u)$ was asymptotically of KPP type as $s\to-\infty$. If the reaction term in (1) is the Logistic growth function, then the population model (1) can be reduced to the Fisher's equation

$$u_t(t,x) = du_{xx}(t,x) + u(t,x)\big[r(x-ct) - u(t,x)\big], \qquad t>0, x\in\mathbb{R}. \tag{2}$$

When the suitable habitat is gradually contracting in the sense that $c>0$ and $r(\cdot)$ is continuous, nondecreasing and connecting some negative number to some positive number, Hu and Zou [11] obtained the existence of extinction wave for the special model (2). Fang et al. [6] studied the existence and non-existence of KPP (monotone) wave and pulse wave to the limiting equation (2) derived from an SIS epidemic model in an expanding or contracting habitat.

Species interactions can influence the range sizes of populations directly or indirectly. A significant background problem is ascertaining when and to what extent biotic interactions really matter in determining range limits. It seems most probably that certain biotic interactions, such as obligate mutualism or generalist predation, may be crucial and needed [3,19]. Mutualism is one of the material interactions among species, which is widely existing in biosphere [28,30], especially collaborative efforts in humans to promote social development [35]. For the competitive species, climate change may be disastrous. For example, climate change drives the plants that were inhabiting previously at lower elevations move to high latitudes, which means that competition generate for alpine plants in the future. But for the cooperative species, what will it happen? The outcomes may depend on the Gambling between cooperation and climate change. A recent review elaborated the correlation of spatial structure, cooperation, mobility and cyclic interaction in evolutionary game [38], see also [24].

Consider a pair of mutualists with densities denoted by u_1 and u_2 and both two species follow the Logistic growth rate which is "on the move" to capture the key point that the environment is both heterogeneous and directionally shifting over time with a forced rate $c>0$. The cooperative behaviors are described by $a_iu_1u_2$ with positive constants a_i for $i=1,2$. Then we can have the following celebrated Lotka–Volterra cooperative system

$$\begin{aligned}\partial_t u_1(t,x) &= d_1\partial_{xx}u_1(t,x) + u_1[r_1(x-ct) - u_1 + a_1u_2],\\ \partial_t u_2(t,x) &= d_2\partial_{xx}u_2(t,x) + u_2[r_2(x-ct) - u_2 + a_2u_1],\end{aligned} \tag{3}$$

where d_1, $d_2>0$ are the diffusion rates. Recently, Lotka–Volterra like models have been well used in metapopulation system under heterogeneous environment such as cyclical interactions with alliance-specific heterogeneous invasion rates or stochasticity in evolutionary processes, for example, see [22,23]. A similar two species cooperative model to (3) can be also found in [3] by replacing the mutualistic effects as the Holling-II response function. Note that the cooperative model proposed in [3] has the same dynamics to (3). For the sake of simplicity and since the dynamics of (3) is sufficient to capture the main interesting properties of the spatial structure of the fronts, hereafter we only consider the traveling waves to (3).

For the existence of traveling wave fronts to some kinds of cooperative systems without considering the climate change, the reader can consult for instance [12,14–18,27,41,42] and references therein. With respect to the diffusion-cooperative system, one might consider that the transition from the origin to the stable state gives rise to the formation of traveling fronts as observed in the Fisher-KPP equation [13]. Furthermore, fronts that initiate from a local perturbation and propagate into a linearly unstable state come in two types: pulled fronts and pushed fronts [5,9,32,34], which extend the previous definitions on general transition waves. Such scenarios commonly appear in monostable reaction-diffusion systems with constant coefficients. For spatial lattice, Szolnoki et al. [36,37] found rich dynamics such as continuous and discontinuous transition waves as well as cyclic dominance happened in human cooperation model. As for the spatial-temporal models with varying coefficients, for example, like the system (3), such topics can still be true?

In this paper, we will study the forced traveling wave fronts of (3) connecting the trivial equilibrium and positive equilibrium, including the existence of fronts and their asymptotics in two tails. Such analyses may provide us some insight on the

stability studies and the category of wave fronts. The results imply that for any given positive speed of the shifting habitat edge, there exists a wave front with the speed consistent with the habitat shifting speed and the wave front decays exponentially in two tails. The wave speed to ensure the existence of front is not determined by the linearized problem around the unstable equilibrium (0,0), which may suggests that it is not linearly determined [9,33]. Such characteristics is novel in cooperative system since this type of monotone system with constant coefficients always satisfies the linear determinacy [41]. We assume

(A1) $r_i(x):\mathbb{R}\to\mathbb{R}$ is continuous, nondecreasing with $-\infty<L_i=r_i(-\infty)<0<r_i(+\infty)=K_i<+\infty,\ i=1,2$;
(A2) $a_1a_2<1, L_1(1-a_1a_2)+a_1(K_2+a_2K_1)<0$ and $L_2(1-a_1a_2)+a_2(K_1+a_1K_2)<0$;
(A3) $r_i(x)$ is continuously differentiable in $\mathbb{R}$ and both $r_i'(\pm\infty)$ exist, $i=1,2$.

Note that the reaction system of (3) admits four equilibria $E_0(0,0)$, $E_1(K_1,0)$, $E_2(0,K_2)$ and $E_*(u_1^*,u_2^*)$, where

$$u_1^*=\frac{K_1+a_1K_2}{1-a_1a_2}>K_1,\ u_2^*=\frac{K_2+a_2K_1}{1-a_1a_2}>K_2.$$

The rest of this article is organized as follows. In Section 2, by constructing appropriate upper and lower solutions to the coupled equations (8) and combined with the classic monotone iteration approaches developed in Wu and Zou [42], we can obtain the existence of traveling wave fronts. In Section 3, we study the asymptotical behavior of traveling wave fronts in two tails. The numerical simulation is given to illustrate our results in the last section.

2. Existence of traveling wave fronts

In this section, we always assume that (A1) and (A2) hold. For simplicity, denote

$$\begin{aligned} f_1(x,u_1,u_2)&=u_1[r_1(x)-u_1+a_1u_2],\\ f_2(x,u_1,u_2)&=u_2[r_2(x)-u_2+a_2u_1]. \end{aligned}\tag{4}$$

For any $0\le u_1,v_1\le u_1^*, 0\le u_2,v_2\le u_2^*$ and $x\in\mathbb{R}$, we have

$$\begin{aligned} |f_1(x,u_1,u_2)-f_1(x,v_1,v_2)|&\le\rho_1\big[|u_1-v_1|+|u_2-v_2|\big],\\ |f_2(x,u_1,u_2)-f_2(x,v_1,v_2)|&\le\rho_2\big[|u_1-v_1|+|u_2-v_2|\big], \end{aligned}\tag{5}$$

where $\rho_i=2u_i^*-L_i+a_i\big(u_1^*+u_2^*\big)$, which imply that $f_1(x, u_1, u_2)$, $f_2(x, u_1, u_2)$ are Lipschitz continuous in $(u_1,u_2)\in[0,u_1^*]\times[0,u_2^*]$ for any $x\in\mathbb{R}$. Define

$$F_i(x,u_1,u_2)=\rho_iu_i+f_i(x,u_1,u_2).\tag{6}$$

Then $F_i(x, u_1, u_2)$ is nondecreasing in $u_i\in[0,u_i^*]$ for $i=1,2$. Let $\mathcal{C}=C(\mathbb{R})$ be all continuous functions from $\mathbb{R}$ to $\mathbb{R}$ and $\mathcal{BC}=\mathcal{C}\cap\mathcal{L}^\infty(\mathbb{R})$ be all continuous and bounded functions from $\mathbb{R}$ to $\mathbb{R}$. Denote $\mathcal{Y}=\mathcal{BC}\times\mathcal{BC}$. For $u=(u_1,u_2)\in\mathcal{Y}$, employ the norm $\|u\|_{\mathcal{Y}}=|u_1|_{\mathcal{BC}}+|u_2|_{\mathcal{BC}}$ with $|u_i|_{\mathcal{BC}}=\sup_{x\in\mathbb{R}}|u_i(x)|$. Then it follows from [21, Theorem 2.1] that the following conclusion holds.

Theorem 2.1. *Consider the Cauchy problem*

$$\begin{aligned} &\partial_tu_1(t,x)=d_1\partial_{xx}u_1(t,x)-\rho_1u_1+F_1(x-ct,u_1,u_2), && t>0,x\in\mathbb{R},\\ &\partial_tu_2(t,x)=d_2\partial_{xx}u_2(t,x)-\rho_2u_2+F_2(x-ct,u_1,u_2), && t>0,x\in\mathbb{R},\\ &u(0,x):=(u_1(0,x),u_2(0,x))=(u_{10}(x),u_{20}(x))=:u_0(x), && x\in\mathbb{R}, \end{aligned}\tag{7}$$

where F_i is defined in (6). If $u_0(x)\in\mathcal{Y}$ with $0\le u_{i0}(x)\le u_i^*$ in $\mathbb{R}$, then (7) has a unique classic solution $u(t,\cdot)\in\mathcal{Y}$ with $0\le u_i(t,x)\le u_i^*$ for all $t>0$ and $x\in\mathbb{R}$.

Now putting $u_i(t,x)=U_i(\xi)$ with $\xi=x-ct$ and plugging it into (3), we have

$$\begin{aligned} &d_1U_1''(\xi)+cU_1'(\xi)+U_1(\xi)[r_1(\xi)-U_1(\xi)+a_1U_2(\xi)]=0,\\ &d_2U_2''(\xi)+cU_2'(\xi)+U_2(\xi)[r_2(\xi)-U_2(\xi)+a_2U_1(\xi)]=0. \end{aligned}\tag{8}$$

The solution of (8) will be considered and which is subjected to the following asymptotic boundary conditions:

$$\lim_{\xi\to-\infty}(U_1(\xi),U_2(\xi))=(0,0),\quad \lim_{\xi\to+\infty}(U_1(\xi),U_2(\xi))=(u_1^*,u_2^*).\tag{9}$$

To construct the proper lower solution of (8), we introduce the following lemma, which can be found in [7]. For vector functions in $\mathbb{R}^n$, we will employ the norm $\|f\|=\sum_{i=1}^n|f_i|$. For the $n\times n$ matrix A with elements a_{ij}, we use the norm $\|A\|=\sum_{i,j=1}^n|a_{ij}|$.

Lemma 2.1. *(Poincaré–Lyapunov). Consider the equation in $\mathbb{R}^n$*

$$\dot{x}=Ax+B(t)x+P(t,x),\quad x(t_0)=x_0,\quad t\in\mathbb{R}.$$

A is a constant $n\times n$ matrix with eigenvalues which have all negative real part; $B(t)$ is a continuous $n\times n$ matrix with the property $\lim_{t\to+\infty}\|B(t)\|=0$. The vector function $P(t, x)$ is continuous in t and x and Lipschitz continuous in x in a neighbourhood of $x=0$; moreover we have

$$\lim_{\|x\|\to 0}\frac{\|P(t,x)\|}{\|x\|}=0 \qquad uniformly\ in\ t.$$

Then there exist constants δ, $\gamma>0$ and $M\ge 1$ such that $\|x_0\|\le\delta$ implies

$$\|x(t)\|\le M\|x_0\|e^{-\gamma(t-t_0)},\quad t\ge t_0.$$

Let $x=(x_1,x_2,x_3,x_4)^T$ with $x_1=U_1$, $x_2=U_1'$, $x_3=U_2$ and $x_4=U_2'$. Then (8) can be rewritten as the following first-order differential equations

$$\begin{aligned}
x_1'(\xi)&=x_2,\\
x_2'(\xi)&=-\frac{c}{d_1}x_2-\frac{1}{d_1}r_1(\xi)x_1+\frac{1}{d_1}x_1^2-\frac{a_1}{d_1}x_1x_3,\\
x_3'(\xi)&=x_4,\\
x_4'(\xi)&=-\frac{c}{d_2}x_4-\frac{1}{d_2}r_2(\xi)x_3+\frac{1}{d_2}x_3^2-\frac{a_2}{d_2}x_1x_3.
\end{aligned}\tag{10}$$

The corresponding vector form of (10) with initial data $x(\xi_0)=x_0$ reads

$$x'=Ax+B(\xi)x+P(x),\quad x(\xi_0)=x_0,\quad \xi\in\mathbb{R},$$

where

$$A=\begin{bmatrix}0&1&0&0\\ -\frac{K_1}{d_1}&-\frac{c}{d_1}&0&0\\ 0&0&0&1\\ 0&0&-\frac{K_2}{d_2}&-\frac{c}{d_2}\end{bmatrix},\quad B(\xi)=\begin{bmatrix}0&0&0&0\\ \frac{1}{d_1}[K_1-r_1(\xi)]&0&0&0\\ 0&0&0&0\\ 0&0&\frac{1}{d_2}[K_2-r_2(\xi)]&0\end{bmatrix},$$

and $P(x)=(0,\frac{1}{d_1}x_1(x_1-a_1x_3),0,\frac{1}{d_2}x_3(x_3-a_2x_1))^T$.

It is easy to check that A, $B(\cdot)$, $P(\cdot)$ satisfy the conditions in Lemma 2.1. Thus there exist constants δ, $\gamma>0$ and $M\ge 1$ such that $\|x_0\|\le\delta$ implies

$$\left\|x\left(\xi;\xi_0,x_0\right)\right\|\le M\|x_0\|e^{-\gamma(\xi-\xi_0)},\quad \xi\ge\xi_0.\tag{11}$$

Here $x\left(\xi;\xi_0,x_0\right)=\left(x_1(\xi;\xi_0,x_0),x_2(\xi;\xi_0,x_0),x_3(\xi;\xi_0,x_0),x_4(\xi;\xi_0,x_0)\right)^T$ with $x\left(\xi_0;\xi_0,x_0\right)=x_0$.

Lemma 2.2. *Let η_i be the positive root of $d_i\lambda^2+c\lambda+L_i=0$ with $L_i=r_i(-\infty)<0$. Then there exit $\alpha_i>\eta_i$ and small $\epsilon>0$ such that the functions*

$$\underline{U}_1(\xi)=\begin{cases}\epsilon e^{\alpha_1(\xi-\xi_0)}, & \xi<\xi_0,\\ x_1\left(\xi;\xi_0,x_0\right), & \xi\ge\xi_0,\end{cases}\quad \underline{U}_2(\xi)=\begin{cases}\epsilon e^{\alpha_2(\xi-\xi_0)}, & \xi<\xi_0,\\ x_3\left(\xi;\xi_0,x_0\right), & \xi\ge\xi_0\end{cases}$$

with $x_0=\epsilon(1,\alpha_1,1,\alpha_2)^T$ are continuously differentiable in $\mathbb{R}$ and satisfy $0<\underline{U}_i(\xi)\le u_i^*$ for all $\xi\in\mathbb{R}$ and $i=1,2$. Moreover, there holds

$$\begin{aligned}
&d_1\underline{U}_1''(\xi)+c\underline{U}_1'(\xi)+\underline{U}_1(\xi)[r_1(\xi)-\underline{U}_1(\xi)+a_1\underline{U}_2(\xi)]\ge 0,\\
&d_2\underline{U}_2''(\xi)+c\underline{U}_2'(\xi)+\underline{U}_2(\xi)[r_2(\xi)-\underline{U}_2(\xi)+a_2\underline{U}_1(\xi)]\ge 0
\end{aligned}\tag{12}$$

for any $\xi\ne\xi_0$.

Proof. Define

$$\epsilon=\min_{i=1,2}\left\{3d_i\eta_i^2+c\eta_i,\ \frac{u_i^*}{2M(1+\eta_1+\eta_2)}\right\},$$

where $M\ge 1$ is defined in (11). Let α_i be the positive root of the equation $d_i\lambda^2+c\lambda+L_i-\epsilon=0$ with $i=1,2$. Then it is easy to check that $\eta_i<\alpha_i\le 2\eta_i$. Since $\lim_{\xi\to\xi_0^-}\underline{U}_1'(\xi)=\epsilon\alpha_1=x_2(\xi_0;\xi_0,x_0)=\lim_{\xi\to\xi_0^+}x_1'(\xi;\xi_0,x_0)$ and $\lim_{\xi\to\xi_0^-}\underline{U}_2'(\xi)=\epsilon\alpha_2=x_4(\xi_0;\xi_0,x_0)=\lim_{\xi\to\xi_0^+}x_3'(\xi;\xi_0,x_0)$, which imply $\underline{U}_1(\xi)$ and $\underline{U}_2(\xi)$ are continuously differentiable in $\mathbb{R}$. Furthermore, it follows from (4), (5) and the uniqueness of solutions to (10) subjected to the positive initial data x_0 that $\underline{U}_i(\xi)>0$ with $i=1,2$.

If $\xi<\xi_0$, then $\underline{U}_i(\xi)=\epsilon e^{\alpha_i(\xi-\xi_0)}<\epsilon\le u_i^*$. Also, for $i\ne j\in\{1, 2\}$, we get

$$d_i\underline{U}_i''(\xi)+c\underline{U}_i'(\xi)+\underline{U}_i(\xi)[r_i(\xi)-\underline{U}_i(\xi)+a_i\underline{U}_j(\xi)]$$

$$\begin{aligned}&\geq d_i\underline{U}_i''(\xi)+c\underline{U}_i'(\xi)+\underline{U}_i(\xi)[r_i(\xi)-\underline{U}_i(\xi)]\\&=\epsilon e^{\alpha_i(\xi-\xi_0)}[d_i\alpha_i^2+c\alpha_i+r_i(\xi)-\epsilon e^{\alpha_i(\xi-\xi_0)}]\\&\geq \epsilon e^{\alpha_i(\xi-\xi_0)}[d_i\alpha_i^2+c\alpha_i+L_i-\epsilon]=0.\end{aligned}$$

If $\xi>\xi_0$, then it follows from (11) that $\underline{U}_1(\xi)=x_1\big(\xi;\xi_0,x_0\big)\leq M\|x_0\|\leq 2M\epsilon(1+\eta_1+\eta_2)\leq u_1^*$. Similarly, $\underline{U}_2(\xi)\leq u_2^*$. Moreover,

$$\begin{aligned}&d_1\underline{U}_1''(\xi)+c\underline{U}_1'(\xi)+\underline{U}_1(\xi)[r_1(\xi)-\underline{U}_1(\xi)+a_1\underline{U}_2(\xi)]\\&=d_1x_2'+cx_2+x_1[r_1(\xi)-x_1+a_1x_3]=0,\end{aligned}$$

and

$$\begin{aligned}&d_2\underline{U}_2''(\xi)+c\underline{U}_2'(\xi)+\underline{U}_2(\xi)[r_2(\xi)-\underline{U}_2(\xi)+a_2\underline{U}_1(\xi)]\\&=d_2x_4'+cx_4+x_3[r_2(\xi)-x_3+a_2x_1]=0.\end{aligned}$$

□

Lemma 2.3. *Let $\xi_1,\ \xi_2\leq\xi_0$ so small that satisfy $r_1(\xi_1)+a_1u_2^*<0$ and $r_2(\xi_2)+a_2u_1^*<0$. Assume β_1 is the positive root of $d_1\lambda^2+c\lambda+r_1(\xi_1)+a_1u_2^*=0$ and β_2 the positive root of $d_2\lambda^2+c\lambda+r_2(\xi_2)+a_2u_1^*=0$. Then the functions $\bar{U}_i(\xi)=\min\left\{u_i^*e^{\beta_i(\xi-\xi_i)},u_i^*\right\}\geq\underline{U}_i(\xi)$ with $i=1,2$ and further satisfy*

$$d_1\bar{U}_1''(\xi)+c\bar{U}_1'(\xi)+\bar{U}_1(\xi)[r_1(\xi)-\bar{U}_1(\xi)+a_1\bar{U}_2(\xi)]\leq 0, \tag{13}$$

for any $\xi\neq\xi_1$;

$$d_2\bar{U}_2''(\xi)+c\bar{U}_2'(\xi)+\bar{U}_2(\xi)[r_2(\xi)-\bar{U}_2(\xi)+a_2\bar{U}_1(\xi)]\leq 0 \tag{14}$$

for any $\xi\neq\xi_2$.

Proof. Note that $r_1(\xi_1)+a_1u_2^*<0$ and $r_2(\xi_2)+a_2u_1^*<0$ can be guaranteed since $r_1(-\infty)(1-a_1a_2)+a_1(K_2+a_2K_1)<0$ and $r_2(-\infty)(1-a_1a_2)+a_2(K_1+a_1K_2)<0$. If $\xi>\xi_1$, then $\bar{U}_1(\xi)=u_1^*\geq\underline{U}_1(\xi)$. Also,

$$\begin{aligned}&d_1\bar{U}_1''(\xi)+c\bar{U}_1'(\xi)+\bar{U}_1(\xi)[r_1(\xi)-\bar{U}_1(\xi)+a_1\bar{U}_2(\xi)]\\&\leq u_1^*[r_1(\xi)-u_1^*+a_1u_2^*]\\&\leq u_1^*[K_1-u_1^*+a_1u_2^*]=0.\end{aligned}$$

If $\xi<\xi_1\leq\xi_0$, then $\bar{U}_1(\xi)=u_1^*e^{\beta_1(\xi-\xi_1)}\geq\epsilon e^{\beta_1(\xi-\xi_0)}\geq\epsilon e^{\alpha_1(\xi-\xi_0)}=\underline{U}_1(\xi)$ due to $\beta_1<\alpha_1$. Also,

$$\begin{aligned}&d_1\bar{U}_1''(\xi)+c\bar{U}_1'(\xi)+\bar{U}_1(\xi)[r_1(\xi)-\bar{U}_1(\xi)+a_1\bar{U}_2(\xi)]\\&\leq u_1^*e^{\beta_1(\xi-\xi_1)}\left[d_1\beta_1^2+c\beta_1+r_1(\xi)-u_1^*e^{\beta_1(\xi-\xi_1)}+a_1u_2^*\right]\\&\leq u_1^*e^{\beta_1(\xi-\xi_1)}[d_1\beta_1^2+c\beta_1+r_1(\xi_1)+a_1u_2^*]=0.\end{aligned}$$

Similarly, we can show that $\bar{U}_2(\xi)\geq\underline{U}_2(\xi)$ and (14) holds for any $\xi\neq\xi_2$. □

Define

$$\Gamma=\left\{(U_1,U_2)\in\mathcal{Y}:(\underline{U}_1,\underline{U}_2)\leq(U_1,U_2)\leq(\bar{U}_1,\bar{U}_2)\text{ in }\mathbb{R}\right\},$$

and

$$\begin{aligned}F_1(U_1,U_2)(\xi)&=\rho_1U_1(\xi)+U_1(\xi)[r_1(\xi)-U_1(\xi)+a_1U_2(\xi)],\\F_2(U_1,U_2)(\xi)&=\rho_2U_2(\xi)+U_2(\xi)[r_2(\xi)-U_2(\xi)+a_2U_1(\xi)].\end{aligned}$$

Then $F=(F_1,F_2)$ is nondecreasing in $U=(U_1,U_2)\in\Gamma$. Rewrite the system (8) as

$$\begin{aligned}&d_1U_1''(\xi)+cU_1'(\xi)-\rho_1U_1(\xi)+F_1(U_1,U_2)(\xi)=0,\\&d_2U_2''(\xi)+cU_2'(\xi)-\rho_2U_2(\xi)+F_2(U_1,U_2)(\xi)=0.\end{aligned}\tag{15}$$

Note that U is a bounded entire solution of (15) if and only if U is continuous and satisfies the integral equation $U=Q(U)$, where $Q=(Q_1,Q_2)$,

$$Q_i(U)(\xi)=\frac{1}{d_i(\lambda_{i2}-\lambda_{i1})}\int_{-\infty}^{+\infty}J_i(\xi-s)F_i(U)(s)ds,$$

and

$$J_i(\xi)=\begin{cases}e^{\lambda_{i2}\xi}, & \xi\leq 0,\\ e^{\lambda_{i1}\xi}, & \xi>0,\end{cases}\quad \lambda_{ij}=\frac{-c+(-1)^j\sqrt{c^2+4d_i\rho_i}}{2d_i},\ i,j=1,2. \tag{16}$$

Lemma 2.4. *The operator Q is nondecreasing and maps Γ into Γ. Moreover, if $U \in \Gamma$ is nondecreasing in ξ, then $Q(U)(\xi)$ is also nondecreasing in ξ.*

Proof. For any V, $W \in \Gamma$ with $V \geq W$, recalling $0 \leq V_i, W_i \leq u_i^*$, we obtain

$$\begin{aligned} &F_1(V_1, V_2)(\xi) - F_1(W_1, W_2)(\xi) \\ &= [\rho_1 + r_1(\xi) - V_1 - W_1 + a_1 V_2][V_1 - W_1] + a_1 W_1 [V_2 - W_2] \geq 0. \end{aligned}$$

In a similar way, F_2 is nondecreasing in U. Hence, by the nonnegativity of J_i defined in (16), we see that

$$\begin{aligned} &Q_i(V)(\xi) - Q_i(W)(\xi) \\ &= \frac{1}{d_i(\lambda_{i2} - \lambda_{i1})} \int_{-\infty}^{+\infty} J_i(\xi - s)\big[F_i(V)(s) - F_i(W)(s)\big] ds \geq 0. \end{aligned}$$

The above inequality implies that

$$Q(\underline{U})(\xi) \leq Q(U)(\xi) \leq Q(\bar{U})(\xi) \tag{17}$$

for all $U \in \Gamma$ and $\xi \in \mathbb{R}$. We now show that Q maps Γ into Γ. For $\xi \neq \xi_0$, without loss of generality, we assume that $\xi < \xi_0$, then by (12), we have

$$\begin{aligned} Q_i(\underline{U})(\xi) &= \frac{1}{d_i(\lambda_{i2} - \lambda_{i1})} \left[\int_{-\infty}^{\xi} e^{\lambda_{i1}(\xi - s)} F_i(\underline{U})(s) ds + \int_{\xi}^{+\infty} e^{\lambda_{i2}(\xi - s)} F_i(\underline{U})(s) ds \right] \\ &\geq \frac{1}{d_i(\lambda_{i2} - \lambda_{i1})} \left(\int_{-\infty}^{\xi} e^{\lambda_{i1}(\xi - s)} + \int_{\xi}^{\xi_0} e^{\lambda_{i2}(\xi - s)} + \int_{\xi_0}^{+\infty} e^{\lambda_{i2}(\xi - s)} \right) \\ &\quad \times \big[-d_i \underline{U}_i''(s) - c\underline{U}_i'(s) + \rho_i \underline{U}_i(s) \big] ds \\ &= \frac{1}{d_i(\lambda_{i2} - \lambda_{i1})} \Big\{ d_i(\lambda_{i2} - \lambda_{i1}) \underline{U}_i(\xi) + d_i e^{\lambda_{i2}(\xi - \xi_0)} [\underline{U}_i'(\xi_0 + 0) - \underline{U}_i'(\xi_0 - 0)] \\ &\quad + (d_i \lambda_{i2} + c) e^{\lambda_{i2}(\xi - \xi_0)} [\underline{U}_i(\xi_0 + 0) - \underline{U}_i(\xi_0 - 0)] \Big\}. \end{aligned}$$

For $\xi > \xi_0$, we get

$$\begin{aligned} Q_i(\underline{U})(\xi) &\geq \frac{1}{d_i(\lambda_{i2} - \lambda_{i1})} \left(\int_{-\infty}^{\xi_0} e^{\lambda_{i1}(\xi - s)} + \int_{\xi_0}^{\xi} e^{\lambda_{i1}(\xi - s)} + \int_{\xi}^{+\infty} e^{\lambda_{i2}(\xi - s)} \right) \\ &\quad \times \big[-d_i \underline{U}_i''(s) - c\underline{U}_i'(s) + \rho_i \underline{U}_i(s) \big] ds \\ &= \frac{1}{d_i(\lambda_{i2} - \lambda_{i1})} \Big\{ d_i(\lambda_{i2} - \lambda_{i1}) \underline{U}_i(\xi) + d_i e^{\lambda_{i1}(\xi - \xi_0)} [\underline{U}_i'(\xi_0 + 0) - \underline{U}_i'(\xi_0 - 0)] \\ &\quad + (d_i \lambda_{i1} + c) e^{\lambda_{i1}(\xi - \xi_0)} [\underline{U}_i(\xi_0 + 0) - \underline{U}_i(\xi_0 - 0)] \Big\}. \end{aligned}$$

Since $\underline{U}_i(\xi)$ is continuously differentiable in ξ_0, we immediately have $Q_i(\underline{U})(\xi) \geq \underline{U}_i(\xi)$ for $\xi \neq \xi_0$. By the continuity, it follows that $Q_i(\underline{U})(\xi) \geq \underline{U}_i(\xi)$ for all ξ.

In a similar way, by using (13) and (14) in Lemma 2.3 and noticing $\bar{U}_i(\xi_i + 0) = \bar{U}_i(\xi_i - 0), \bar{U}_i'(\xi_i + 0) \leq \bar{U}_i'(\xi_i - 0)$, we can also prove that $Q_i(\bar{U})(\xi) \leq \bar{U}_i(\xi)$ for $\xi \neq \xi_i$. Therefore, the continuity of Q implies $Q_i(\bar{U})(\xi) \leq \bar{U}_i(\xi)$ for all ξ. These, together with (17), we see that Q maps Γ into Γ.

If $U \in \Gamma$ is nondecreasing in ξ, then for any $\xi \in \mathbb{R}$ and $\theta > 0$, we get

$$\begin{aligned} &F_i(U)(\xi + \theta) - F_i(U)(\xi) \\ &= \big[\rho_i + r_i(\xi) - U_i(\xi + \theta) - U_i(\xi) + a_i U_j(\xi + \theta)\big][U_i(\xi + \theta) - U_i(\xi)] \\ &\quad + [r_i(\xi + \theta) - r_i(\xi)] U_i(\xi + \theta) + a_i U_i(\xi)[U_j(\xi + \theta) - U_j(\xi)] \geq 0 \end{aligned}$$

for $i \neq j \in \{1, 2\}$. Hence,

$$\begin{aligned} Q_i(U)(\xi + \theta) &= \frac{1}{d_i(\lambda_{i2} - \lambda_{i1})} \int_{-\infty}^{+\infty} J_i(\xi + \theta - s) F_i(U)(s) ds \\ &= \frac{1}{d_i(\lambda_{i2} - \lambda_{i1})} \int_{-\infty}^{+\infty} J_i(\xi - s) F_i(U)(s + \theta) ds \\ &\geq \frac{1}{d_i(\lambda_{i2} - \lambda_{i1})} \int_{-\infty}^{+\infty} J_i(\xi - s) F_i(U)(s) ds = Q_i(U)(\xi). \end{aligned}$$

□

Theorem 2.2. *Assume (A1) and (A2) hold. Then (3) has a nondecreasing traveling wave front $(u_1(t, x), u_2(t, x)) = (U_1(\xi), U_2(\xi))$ with $\xi = x - ct$.*

Proof. Define the following iterations:

$$U_i^{(1)} = Q_i(\bar{U}_1, \bar{U}_2), \quad U_i^{(k+1)} = Q_i\big(U_1^{(k)}, U_2^{(k)}\big), \quad k \geq 1.$$

Since $(\bar{U}_1, \bar{U}_2) \in \Gamma$ is nondecreasing in $\mathbb{R}$, Lemma 2.4 implies $(U_1^{(k)}, U_2^{(k)}) \in \Gamma$, and $(U_1^{(k)}, U_2^{(k)})$ is nondecreasing in ξ for each fixed $k = 1, 2, \ldots,$ and further satisfies

$$(\underline{U}_1(\xi), \underline{U}_2(\xi)) \leq (U_1^{(k+1)}(\xi), U_2^{(k+1)}(\xi)) \leq (U_1^{(k)}(\xi), U_2^{(k)}(\xi)) \leq (\bar{U}_1(\xi), \bar{U}_2(\xi))$$

for all $k = 1, 2, \ldots$ and $\xi \in \mathbb{R}$. Thus $\lim_{k\to\infty} U_i^{(k)}(\xi) = U_i(\xi)$ both exist for $i = 1, 2$. Obviously, $U_i(\xi)$ is nondecreasing and nonnegative in $\xi \in \mathbb{R}$. Moreover, we have the following properties

$$(\underline{U}_1(\xi), \underline{U}_2(\xi)) \leq (U_1(\xi), U_2(\xi)) \leq (\bar{U}_1(\xi), \bar{U}_2(\xi)), \qquad \forall\, \xi \in \mathbb{R}. \tag{18}$$

By the continuity of F_i, we know that $F_i\big(U_1^{(k)}, U_2^{(k)}\big)$ respectively converges point-wise to $F_i(U_1, U_2)$. We now show that (U_1, U_2) is a fixed-point of $F = (F_1, F_2)$. Since

$$\Big|F_i\big(U_1^{(k)}, U_2^{(k)}\big)\Big| \leq \rho_i u_i^*,$$

then the Lebesgue's dominated convergence theorem yields that

$$\begin{aligned} U_i(\xi) &= \lim_{k\to\infty} U_i^{(k+1)}(\xi) \\ &= \lim_{k\to\infty} Q_i\big(U_1^{(k)}, U_2^{(k)}\big)(\xi) \\ &= \lim_{k\to\infty} \frac{1}{d_i(\lambda_{i2}-\lambda_{i1})} \int_{-\infty}^{+\infty} J_i(\xi - s) F_i\big(U_1^{(k)}, U_2^{(k)}\big)(s)\mathrm{d}s \\ &= \frac{1}{d_i(\lambda_{i2}-\lambda_{i1})} \int_{-\infty}^{+\infty} J_i(\xi - s) F_i(U_1, U_2)(s)\mathrm{d}s = Q_i(U_1, U_2)(\xi). \end{aligned}$$

Thus (U_1, U_2) is a solution of (15). Next, we prove (U_1, U_2) satisfies the asymptotical boundary conditions (9). Since

$$(\underline{U}_1(-\infty), \underline{U}_2(-\infty)) = (0, 0) = (\bar{U}_1(-\infty), \bar{U}_2(-\infty)),$$

hence by (18), we see that $(U_1(-\infty), U_2(-\infty)) = (0, 0)$. Note that $U_i(\xi)$ is nondecreasing in $\xi \in \mathbb{R}$ and $U_i \in [0, u_i^*]$. Therefore, $\lim_{\xi\to+\infty} U_i(\xi)$ exists, denote it by κ_i, respectively. Also, $\kappa_i \in [0, u_i^*]$ and $\lim_{\xi\to+\infty} F_i(U_1, U_2)(\xi) = \rho_i\kappa_i + \kappa_i(K_i - \kappa_i + a_i\kappa_j)$ for $i \neq j \in \{1, 2\}$. In view of L'Hôpital's rule, we obtain

$$\begin{aligned} \kappa_i &= \lim_{\xi\to+\infty} U_i(\xi) = \lim_{\xi\to+\infty} Q_i(U_1, U_2)(\xi) \\ &= \lim_{\xi\to+\infty} \frac{1}{d_i(\lambda_{i2}-\lambda_{i1})} \left[\int_{-\infty}^{\xi} \mathrm{e}^{\lambda_{i1}(\xi - s)} F_i(U_1, U_2)(s)\mathrm{d}s + \int_{\xi}^{+\infty} \mathrm{e}^{\lambda_{i2}(\xi - s)} F_i(U_1, U_2)(s)\mathrm{d}s\right] \\ &= \frac{1}{d_i(\lambda_{i2}-\lambda_{i1})} \left[\frac{\rho_i\kappa_i + \kappa_i(K_i - \kappa_i + a_i\kappa_j)}{-\lambda_{i1}} + \frac{\rho_i\kappa_i + \kappa_i(K_i - \kappa_i + a_i\kappa_j)}{\lambda_{i2}}\right] \\ &= \frac{1}{\rho_i}\big[\rho_i\kappa_i + \kappa_i(K_i - \kappa_i + a_i\kappa_j)\big] = \kappa_i + \frac{\kappa_i(K_i - \kappa_i + a_i\kappa_j)}{\rho_i}. \end{aligned}$$

Thus, $\kappa_i(K_i - \kappa_i + a_i\kappa_j) = 0$ for $i \neq j \in \{1, 2\}$. Since $\kappa_i \in [0, u_i^*]$, it must be $\kappa_i = 0$ or $K_i - \kappa_i + a_i\kappa_j = 0$. If $\kappa_i = 0$, then since U_i is nondecreasing in $\mathbb{R}$ and $U_i(\pm\infty) = 0$, we know that $U_i \equiv 0$ in $\mathbb{R}$. This contradicts $U_i(\xi) \geq \underline{U}_i(\xi)$ and the definition of $\underline{U}_i$. Thus, $K_i - \kappa_i + a_i\kappa_j = 0$, which is equivalent to $K_1 - \kappa_1 + a_1\kappa_2 = 0$ and $K_2 - \kappa_2 + a_2\kappa_1 = 0$. Some trivial calculations show that $\kappa_1 = u_1^*$ and $\kappa_2 = u_2^*$. The asymptotic boundary conditions (9) are satisfied. □

3. Asymptotic behaviors of traveling waves

To obtain more asymptotic information of the traveling waves at $\pm\infty$, we let

$$(U_1(\xi), U_2(\xi)) = (u_1^0(\xi), u_2^0(\xi)) \qquad \text{for } -\infty < \xi < +\infty,$$

which is a solution of (8) satisfies (9). Differentiating (8) with respect to ξ, then we know that $(U_1'(\xi), U_2'(\xi)) =: (\omega_1, \omega_2)$ satisfies

$$\begin{aligned} &d_1\omega_1''(\xi) + c\omega_1'(\xi) + [r_1(\xi) - 2u_1^0 + a_1u_2^0]\omega_1 + a_1u_1^0\omega_2 + r_1'(\xi)u_1^0 = 0, \\ &d_2\omega_2''(\xi) + c\omega_2'(\xi) + [r_2(\xi) - 2u_2^0 + a_2u_1^0]\omega_2 + a_2u_2^0\omega_1 + r_2'(\xi)u_2^0 = 0. \end{aligned} \tag{19}$$

Theorem 3.1. *Assume (A1) to (A3) hold. Then there exist positive constants A_i, B_i, C_i with $i = 1, 2$ such that the traveling wave front $(U_1(\xi), U_2(\xi))$ to (3) has the following asymptotic properties*

$$\begin{pmatrix} U_1(\xi) \\ U_2(\xi) \end{pmatrix} = \begin{pmatrix} (A_1 + o(1))e^{\frac{-c+\sqrt{c^2-4d_1L_1}}{2d_1}\xi} \\ (A_2 + o(1))e^{\frac{-c+\sqrt{c^2-4d_2L_2}}{2d_2}\xi} \end{pmatrix}$$

as $\xi \to -\infty$; and

$$\begin{pmatrix} U_1(\xi) \\ U_2(\xi) \end{pmatrix} = \begin{pmatrix} u_1^* - (B_1 + o(1))e^{\lambda_2\xi} - (C_1 + o(1))e^{\lambda_4\xi} \\ u_2^* - (B_2 + o(1))e^{\lambda_2\xi} - (C_2 + o(1))e^{\lambda_4\xi} \end{pmatrix},$$

as $\xi \to +\infty$ provided $d_1 = d_2$. Here λ_2, λ_4 are defined in (25).

Proof. The limiting equations for (19) as $\xi \to -\infty$ read

$$\begin{aligned} &d_1\phi_1''(\xi) + c\phi_1'(\xi) + L_1\phi_1(\xi) = 0, \\ &d_2\phi_2''(\xi) + c\phi_2'(\xi) + L_2\phi_2(\xi) = 0. \end{aligned} \tag{20}$$

The first equation of (20) has two independent solutions in the following form

$$\phi_1^1(\xi) = e^{\frac{-c-\sqrt{c^2-4d_1L_1}}{2d_1}\xi}, \quad \phi_1^2(\xi) = e^{\frac{-c+\sqrt{c^2-4d_1L_1}}{2d_1}\xi}.$$

Relating (19) and (20), we see that ω_1 admits the following property as $\xi \to -\infty$

$$\omega_1(\xi) = p_1[1 + o(1)]\phi_1^1(\xi) + q_1[1 + o(1)]\phi_1^2(\xi).$$

Since $\lim_{\xi\to-\infty}\omega_1(\xi) = 0$, it must be $p_1 = 0$. Hence, for $\xi \to -\infty$,

$$U_1'(\xi) = \omega_1(\xi) = q_1[1 + o(1)]e^{\frac{-c+\sqrt{c^2-4d_1L_1}}{2d_1}\xi}.$$

Similarly,

$$U_2'(\xi) = \omega_2(\xi) = q_2[1 + o(1)]e^{\frac{-c+\sqrt{c^2-4d_2L_2}}{2d_2}\xi},$$

as $\xi \to -\infty$. By integration from $-\infty$ to ξ, it follows that

$$\begin{pmatrix} U_1(\xi) \\ U_2(\xi) \end{pmatrix} = \begin{pmatrix} (A_1 + o(1))e^{\frac{-c+\sqrt{c^2-4d_1L_1}}{2d_1}\xi} \\ (A_2 + o(1))e^{\frac{-c+\sqrt{c^2-4d_2L_2}}{2d_2}\xi} \end{pmatrix}$$

as $\xi \to -\infty$ for some constants $A_1, A_2 > 0$. Now the limiting equations for (19) as $\xi \to +\infty$ are

$$\begin{aligned} &d_1\psi_1''(\xi) + c\psi_1'(\xi) + A\psi_1(\xi) + B\psi_2(\xi) = 0, \\ &d_2\psi_2''(\xi) + c\psi_2'(\xi) + C\psi_1(\xi) + D\psi_2(\xi) = 0, \end{aligned} \tag{21}$$

where

$$A = K_1 - 2u_1^* + a_1u_2^* = -u_1^* < 0, \quad B = a_1u_1^* > 0,$$

$$D = K_2 - 2u_2^* + a_2u_1^* = -u_2^* < 0, \quad C = a_2u_2^* > 0.$$

Note that in (21) we have used the fact that $r_i'(+\infty) = 0$ with $i = 1, 2$. Indeed, by applying L'Hôpital's rule, we get

$$\lim_{\xi\to+\infty} r_i(\xi) = \lim_{\xi\to+\infty} \frac{e^\xi r_i(\xi)}{e^\xi} = \lim_{\xi\to+\infty} \frac{e^\xi[r_i(\xi) + r_i'(\xi)]}{e^\xi} = \lim_{\xi\to+\infty}[r_i(\xi) + r_i'(\xi)],$$

which implies $r_i'(+\infty) = 0$. Letting $\psi_1' = \tilde{\psi}_1$, $\psi_2' = \tilde{\psi}_2$, then we can rewrite (21) as the following first-order differential equations

$$\begin{aligned} \psi_1'(\xi) &= \tilde{\psi}_1, \\ \tilde{\psi}_1'(\xi) &= -\frac{c}{d_1}\tilde{\psi}_1 - \frac{A}{d_1}\psi_1 - \frac{B}{d_1}\psi_2, \\ \psi_2'(\xi) &= \tilde{\psi}_2, \\ \tilde{\psi}_2'(\xi) &= -\frac{c}{d_2}\tilde{\psi}_2 - \frac{C}{d_2}\psi_1 - \frac{D}{d_2}\psi_2. \end{aligned} \tag{22}$$

The characteristic equation of (22) is

$$\lambda^2\Big(\lambda+\frac{c}{d_1}\Big)\Big(\lambda+\frac{c}{d_2}\Big)+\frac{D}{d_2}\lambda\Big(\lambda+\frac{c}{d_1}\Big)+\frac{A}{d_1}\lambda\Big(\lambda+\frac{c}{d_2}\Big)+\frac{AD-BC}{d_1d_2}=0. \tag{23}$$

If $d_1=d_2=d$, then (23) can be simplified as

$$\lambda^2\Big(\lambda+\frac{c}{d}\Big)^2+\frac{1}{d}(A+D)\lambda\Big(\lambda+\frac{c}{d}\Big)+\frac{1}{d^2}(AD-BC)=0.$$

Let $\lambda\big(\lambda+\frac{c}{d}\big)=y$. Then y satisfies $y^2+\frac{1}{d}(A+D)y+\frac{1}{d^2}(AD-BC)=0$. Since

$$(A+D)^2-4(AD-BC)=(A-D)^2+4BC>0,$$
$$AD=u_1^*u_2^*>a_1a_2u_1^*u_2^*=BC,\quad A+D<0,$$

we have $y_j=\frac{-(A+D)+(-1)^j\sqrt{(A+D)^2-4(AD-BC)}}{2d}>0$ with $j=1,2$. Thus the general solution corresponding to (22) can be expressed as

$$(\psi_1,\tilde{\psi}_1,\psi_2,\tilde{\psi}_2)^T=\ell_1h_1e^{\lambda_1\xi}+\ell_2h_2e^{\lambda_2\xi}+\ell_3h_3e^{\lambda_3\xi}+\ell_4h_4e^{\lambda_4\xi}, \tag{24}$$

where

$$\begin{aligned}\lambda_1&=\frac{-c+\sqrt{c^2+4d^2y_1}}{2d}>0,\quad \lambda_2=\frac{-c-\sqrt{c^2+4d^2y_1}}{2d}<0\\ \lambda_3&=\frac{-c+\sqrt{c^2+4d^2y_2}}{2d}>0,\quad \lambda_4=\frac{-c-\sqrt{c^2+4d^2y_2}}{2d}<0,\end{aligned} \tag{25}$$

and h_i are eigenvectors corresponding to λ_i, ℓ_i are arbitrary constants with $i=1,2,3,4$. Since $(\psi_1,\tilde{\psi}_1,\psi_2,\tilde{\psi}_2)^T\to(0,0,0,0)$ as $\xi\to+\infty$, we follow from (24) that $\ell_1=0$ and $\ell_3=0$. Thus, $(\psi_1,\tilde{\psi}_1,\psi_2,\tilde{\psi}_2)^T=\ell_2h_2e^{\lambda_2\xi}+\ell_4h_4e^{\lambda_4\xi}$. Hence, (ω_1,ω_2) possesses the following property as $\xi\to+\infty$

$$\begin{pmatrix}\omega_1(\xi)\\ \omega_2(\xi)\end{pmatrix}=\begin{pmatrix}\nu_1(m_1+o(1))e^{\lambda_2\xi}+\nu_2(m_2+o(1))e^{\lambda_4\xi}\\ \nu_1(n_1+o(1))e^{\lambda_2\xi}+\nu_2(n_2+o(1))e^{\lambda_4\xi}\end{pmatrix},$$

where m_i, n_i, ν_i are constants and ν_i cannot be zero simultaneously, $i=1,2$. Indeed, $m_i\neq0$ and $n_i\neq0$. For the solution $h_ie^{\lambda_i\xi}$ to (22), if one of the first and third components of the eigenvectors h_i is zero, then the linear system (22) leads to the other components are also zero. By integrating from ξ to $+\infty$, it follows that

$$\begin{pmatrix}U_1(\xi)\\ U_2(\xi)\end{pmatrix}=\begin{pmatrix}u_1^*-(B_1+o(1))e^{\lambda_2\xi}-(C_1+o(1))e^{\lambda_4\xi}\\ u_2^*-(B_2+o(1))e^{\lambda_2\xi}-(C_2+o(1))e^{\lambda_4\xi}\end{pmatrix},$$

as $\xi\to+\infty$ and $d_1=d_2$. □

4. Simulations

Truncate $\mathbb{R}=(-\infty,\infty)$ by $[-X,X]$ for some large X and adopt the uniform partition of $[-X,X]$ as

$$-X=\xi_1<\xi_2<\cdots<\xi_{2n-1}<\xi_{2n}<\xi_{2n+1}=X,$$

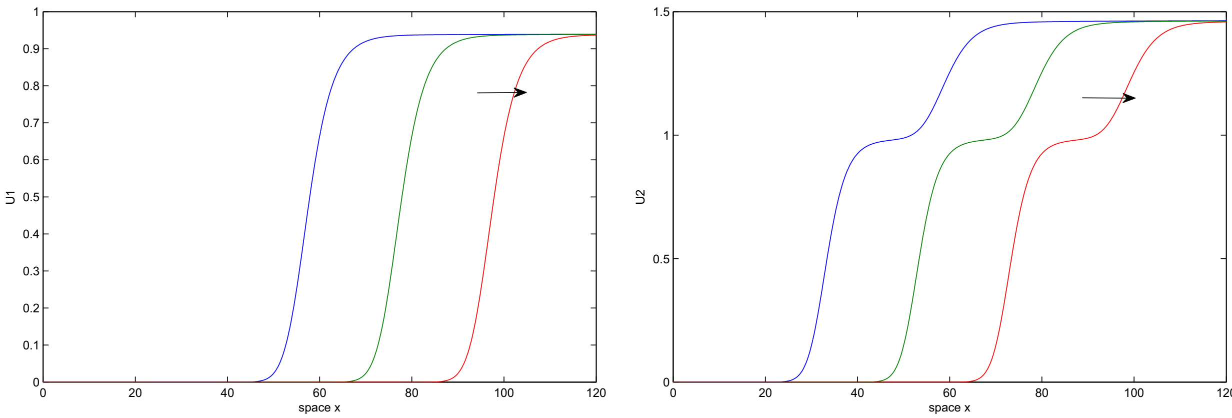

Fig. 1. Illustration of monotone forced wave fronts in the discrete system (26) corresponding to wave profile system (8) with Logistic growth functions $r_1(x)=\frac{2}{1+e^{-x}}-1.5$ and $r_2(x)=\frac{2}{\pi}\arctan x$. Parameters are $a_1=0.3, a_2=0.5, d_1=d_2=1, c=2, h=0.1$. The boundary condition is the characteristic function on $(-\infty,0)$. The wave front of U_1-species is shown in the left plot and U_2-species in the right plot with every 10 time steps.

where $\xi_i = \xi_1 + (i-1)h,\ h = \frac{X}{n}, i = 1, 2, \ldots, 2n+1$. Corresponding to the truncation, then the asymptotic boundary conditions (9) become to

$$(U_1(\xi_1), U_2(\xi_1)) = (0, 0),\ (U_1(\xi_{2n+1}), U_2(\xi_{2n+1})) = (u_1^*, u_2^*).$$

By using the conventional numeric differentiation, we get a discrete system corresponding to (8) as follows

$$\begin{aligned}&(-ch + d_1)U_1(\xi_{i-1}) - (-ch + 2d_1)U_1(\xi_i) + d_1U_1(\xi_{i+1}) + h^2U_1(\xi_i)[r_1(\xi_i) - U_1(\xi_i) + a_1U_2(\xi_i)] = 0,\\&(-ch + d_2)U_2(\xi_{i-1}) - (-ch + 2d_2)U_2(\xi_i) + d_2U_2(\xi_{i+1}) + h^2U_2(\xi_i)[r_2(\xi_i) - U_2(\xi_i) + a_2U_1(\xi_i)] = 0.\end{aligned} \tag{26}$$

Let $r_1(x) = \frac{2}{1+e^{-x}} - 1.5, r_2(x) = \frac{2}{\pi}\arctan x, a_1 = 0.3, a_2 = 0.5, d_1 = d_2 = 1, c = 2, h = 0.1$. Then $(u_1^*, u_2^*) \approx (0.9412, 1.4706)$ and (A1) to (A3) hold. From Fig. 1, we see that the forced wave fronts are exponentially decaying in two tails.

Acknowledgments

The authors would like to thank the associate editor and the referees for their very helpful comments and suggestions.

Supplementary material

Supplementary material associated with this article can be found, in the online version, at doi:10.1016/j.amc.2019.01.058.

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